# LIGO-GW150914 (10 points)

In 2015, the gravitational-wave observatory LIGO detected, for the first time, the passing of gravitational waves (GW) through Earth. This event, named GW150914, was triggered by waves produced by two black holes that were orbiting on quasi-circular orbits. This problem will make you estimate some physical parameters of the system, from the properties of the detected signal.

## Part A: Newtonian (conservative) orbits (3.0 points)

**A.1** Consider a system of two stars with masses $M_1, M_2$, at locations $\vec{r}_1, \vec{r}_2$, respectively, with respect to the center-of-mass of the system, that is, **1.0pt**

$$M_1\vec{r}_1 + M_2\vec{r}_2 = 0\,. \tag{1}$$

The stars are isolated from the rest of the Universe and moving at non-relativistic velocities. Using Newton's laws, the acceleration vector of mass $M_1$ can be expressed as

$$\frac{\mathrm{d}^2\vec{r}_1}{\mathrm{d}t^2} = -\alpha\frac{\vec{r}_1}{r_1^n}\,, \tag{2}$$

where $r_1 = |\vec{r}_1|, r_2 = |\vec{r}_2|$. Find $n \in \mathbb{N}$ and $\alpha = \alpha(G, M_1, M_2)$, where $G$ is Newton's constant [$G \simeq 6.67 \times 10^{-11}$ N m$^2$ kg$^{-2}$].

**A.2** The total energy of the 2-mass system, in circular orbits, can be expressed as: **1.0pt**

$$E = A(\mu, \Omega, L) - G\frac{M\mu}{L}\,, \tag{3}$$

where

$$\mu \equiv \frac{M_1M_2}{M_1+M_2}\,, \qquad M \equiv M_1 + M_2\,, \tag{4}$$

are the *reduced mass* and *total mass* of the system, $\Omega$ is the angular velocity of each mass and $L$ is the total separation $L = r_1 + r_2$. Obtain the explicit form of the term $A(\mu, \Omega, L)$.

**A.3** Equation 3 can be simplified to $E = \beta G\frac{M\mu}{L}$. Determine the number $\beta$. **1.0pt**

## Part B: Introducing relativistic dissipation (7.0 points)

The correct theory of gravity, *General Relativity*, was formulated by Einstein in 1915, and predicts that gravity travels with the speed of light. The messengers carrying information about the interaction are called GWs. GWs are emitted whenever masses are accelerated, making the system of masses lose energy.

Consider a system of two point-like particles, isolated from the rest of the Universe. Einstein proved that for small enough velocities the emitted GWs: 1) have a frequency which is twice as large as the orbital frequency; 2) can be characterized by a luminosity, i.e. emitted power $\mathcal{P}$, which is dominated by Einstein's quadrupole formula,

$$\mathcal{P} = \frac{G}{5c^5}\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3}\right)\left(\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3}\right) . \tag{5}$$

Here, $c$ is the velocity of light $c \simeq 3\times10^8$ m/s. For a system of 2 pointlike particles orbiting on the $x-y$ plane, $Q_{ij}$ is the following table ($i,j$ label the row/column number)

$$Q_{11} = \sum_{A=1}^{2}\frac{M_A}{3}\left(2x_A^2 - y_A^2\right) , \qquad Q_{22} = \sum_{A=1}^{2}\frac{M_A}{3}\left(2y_A^2 - x_A^2\right) , \qquad Q_{33} = -\sum_{A=1}^{2}\frac{M_A}{3}(x_A^2 + y_A^2) , \tag{6}$$

$$Q_{12} = Q_{21} = \sum_{A=1}^{2} M_A\, x_A\, y_A , \tag{7}$$

and $Q_{ij} = 0$ for all other possibilities. Here, $(x_A, y_A)$ is the position of mass A in the center-of-mass frame.

**B.1** For the circular orbits described in A.2 the components of $Q_{ij}$ can be expressed as a function of time $t$ as: 1.0pt

$$Q_{ii} = \frac{\mu L^2}{2}\left(a_i + b_i \cos kt\right) , \qquad Q_{ij} \overset{i\neq j}{=} \frac{\mu L^2}{2}\, c_{ij} \sin kt . \tag{8}$$

Determine $k$ in terms of $\Omega$ and the numerical values of the constants $a_i$, $b_i$, $c_{ij}$.

**B.2** Compute the power $\mathcal{P}$ emitted in gravitational waves for that system, and obtain: 1.0pt

$$\mathcal{P} = \xi \frac{G}{c^5}\mu^2 L^4 \Omega^6 . \tag{9}$$

What is the number $\xi$? [If you could not obtain $\xi$, use $\xi = 6.4$ in the following.]

**B.3** In the absence of GW emission the two masses will orbit on a fixed circular orbit indefinitely. However, the emission of GWs causes the system to lose energy and to slowly evolve towards smaller circular orbits. Obtain that the rate of change $\frac{\mathrm{d}\Omega}{\mathrm{d}t}$ of the orbital angular velocity takes the form 1.0pt

$$\left(\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right)^3 = (3\xi)^3 \frac{\Omega^{11}}{c^{15}}\left(GM_{\mathrm{c}}\right)^5 , \tag{10}$$

where $M_{\mathrm{c}}$ is called the *chirp mass*. Obtain $M_{\mathrm{c}}$ as a function of $M$ and $\mu$. This mass determines the increase in frequency during the orbital decay. [The name "chirp" is inspired by the high pitch sound (increasing frequency) produced by small birds.]

**B.4** Using the information provided above, relate the orbital angular velocity $\Omega$ with the GW frequency $f_{\text{GW}}$. Knowing that, for any smooth function $F(t)$ and $a \neq 1$, 2.0pt

$$\frac{\mathrm{d}F(t)}{\mathrm{d}t} = \chi F(t)^a \qquad \Rightarrow \qquad F(t)^{1-a} = \chi(1-a)(t-t_0)\,, \tag{11}$$

where $\chi$ is a constant and $t_0$ is an integration constant, show that (10) implies that the GW frequency is

$$f_{\text{GW}}^{-8/3} = 8\pi^{8/3}\xi\left(\frac{GM_c}{c^3}\right)^{(2/3)+p}(t_0-t)^{2-p} \tag{12}$$

and determine the constant $p$.

On September 14, 2015 GW150914 was registered by the LIGO detectors, consisting of two L-shaped arms, each 4 km long. These arms changed by a relative length according to Fig. 1. The arms of the detector respond linearly to a passing gravitational wave, and the response pattern mimics the wave. This wave was created by two black holes on quasi-circular orbits; the loss of energy through gravitational radiation caused the orbit to shrink and the black holes to eventually collide. The collision point corresponds, roughly, to the peak of the signal after point D, in Fig. 1.

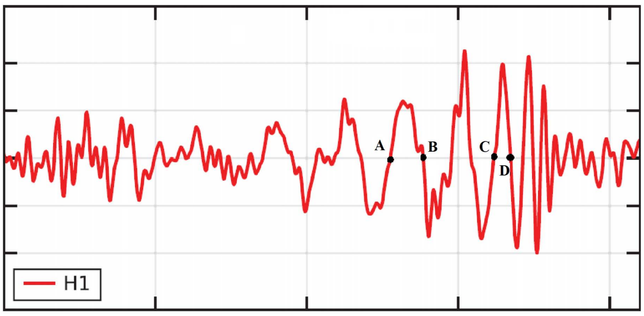


Figure 1. Strain, i.e. relative variation of the size of each arm, at the LIGO detector H1. The horizontal axis is time, and the points A, B, C, D correspond to $t = 0.000, 0.009, 0.034, 0.040$ seconds, respectively.

**B.5** From the figure, estimate $f_{\text{GW}}(t)$ at 1.0pt

$$t_{\overline{\text{AB}}} = \frac{t_{\text{B}}+t_{\text{A}}}{2} \quad \text{and} \quad t_{\overline{\text{CD}}} = \frac{t_{\text{D}}+t_{\text{C}}}{2}\,. \tag{13}$$

Assuming that (12) is valid all the way until the collision (which strictly speaking is not true) and that the two objects have equal mass, estimate the chirp mass, $M_c$, and total mass of the system, in terms of solar masses $M_\odot \simeq 2\times 10^{30}$ kg.

**B.6** Estimate the minimal orbital separation between the two objects at $t_{\overline{\text{CD}}}$. Hence estimate a maximum size for each object, $R_{\text{max}}$. Obtain $R_\odot/R_{\text{max}}$ to compare this size with the radius of our Sun, $R_\odot \simeq 7\times 10^5$ km. Estimate also their orbital linear velocity at the same instant, $v_{\text{col}}$, comparing it with the speed of light, $v_{\text{col}}/c$. 1.0pt

Conclude that these are extremely fast moving, extremely compact objects indeed!

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# Solutions to Theory Problem 1

## LIGO-GW150914

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v6.0

# GW150914 (10 points)

## Part A. Newtonian (conservative) orbits (3.0 points)

**A.1** Apply Newton's law to mass $M_1$:

$$M_1 \frac{\mathrm{d}^2\vec{r}_1}{\mathrm{d}t^2} = G\frac{M_1 M_2}{|\vec{r}_2 - \vec{r}_1|^2} \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|} \ . \tag{1}$$

Use, from eq. (1) of the question sheet

$$\vec{r}_2 = -\frac{M_1}{M_2}\vec{r}_1 \ , \tag{2}$$

in eq. (1) above, to obtain

$$\frac{\mathrm{d}^2\vec{r}_1}{\mathrm{d}t^2} = -\frac{GM_2^3}{(M_1+M_2)^2 r_1^2}\frac{\vec{r}_1}{r_1} \ . \tag{3}$$

| **A.1** | 1.0pt |
|---|---|
| $n = 3, \qquad \alpha = \dfrac{GM_2^3}{(M_1+M_2)^2} \ .$ | |

**A.2** The total energy of the system is the sum of the two kinetic energies plus the gravitational potential energy. For circular motions, the linear velocity of each of the masses reads

$$|\vec{v}_1| = r_1\Omega \ , \qquad |\vec{v}_2| = r_2\Omega \ , \tag{4}$$

Thus, the total energy is

$$E = \frac{1}{2}(M_1 r_1^2 + M_2 r_2^2)\Omega^2 - \frac{GM_1M_2}{L} \ , \tag{5}$$

Now,

$$(M_1 r_1 - M_2 r_2)^2 = 0 \qquad \Rightarrow \qquad M_1 r_1^2 + M_2 r_2^2 = \mu L^2 \ . \tag{6}$$

Thus,

$$E = \frac{1}{2}\mu L^2\Omega^2 - G\frac{M\mu}{L} \ . \tag{7}$$

| **A.2** | 1.0pt |
|---|---|
| $A(\mu, \Omega, L) = \dfrac{1}{2}\mu L^2\Omega^2 \ .$ | |

**A.3** Energy (3) of the question sheet can be interpreted as describing a system of a mass $\mu$ in a circular orbit with angular velocity $\Omega$, radius $L$, around a mass $M$ (at rest). Equating the gravitational acceleration to the centripetal acceleration:

$$G\frac{M}{L^2} = \Omega^2 L \ . \tag{8}$$

This is indeed Kepler's third law (for circular orbits). Then, from (7),

$$E = -\frac{1}{2}G\frac{M\mu}{L} \ . \tag{9}$$

| **A.3** | 1.0pt |
|---|---|
| $\beta = -\dfrac{1}{2} \ .$ | |

## Part B - Introducing relativistic dissipation (7.0 points)

**B.1** Some simple trigonometry for the $x, y$ motion of the masses (in an appropriate Cartesian system) yields:

$$(x_1, y_1) = r_1(\cos(\Omega t), \sin(\Omega t)) \,, \qquad (x_2, y_2) = -r_2(\cos(\Omega t), \sin(\Omega t)) \,. \tag{10}$$

Then,

$$Q_{ij} = \frac{M_1 r_1^2 + M_2 r_2^2}{2} \begin{pmatrix} \frac{4}{3}\cos^2(\Omega t) - \frac{2}{3}\sin^2(\Omega t) & 2\sin(\Omega t)\cos(\Omega t) & 0 \\ 2\sin(\Omega t)\cos(\Omega t) & \frac{4}{3}\sin^2(\Omega t) - \frac{2}{3}\cos^2(\Omega t) & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix} \,, \tag{11}$$

or, using some simple trigonometry and (6),

$$Q_{ij} = \frac{\mu L^2}{2} \begin{pmatrix} \frac{1}{3} + \cos 2\Omega t & \sin 2\Omega t & 0 \\ \sin 2\Omega t & \frac{1}{3} - \cos 2\Omega t & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix} \,. \tag{12}$$

**B.1** 1.0pt

$$k = 2\Omega \,, \quad a_1 = a_2 = \frac{1}{3} \,, a_3 = -\frac{2}{3} \,, \quad b_1 = 1, b_2 = -1, b_3 = 0 \,, c_{12} = c_{21} = 1, c_{ij} \overset{\text{otherwise}}{=} 0 \,.$$

**B.2** First take the derivatives:

$$\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3} = 4\Omega^3 \mu L^2 \begin{pmatrix} \sin 2\Omega t & -\cos 2\Omega t & 0 \\ -\cos 2\Omega t & -\sin 2\Omega t & 0 \\ 0 & 0 & 0 \end{pmatrix} \,. \tag{13}$$

Then perform the sum:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{G}{5c^5}(4\Omega^3 \mu L^2)^2 [2\sin^2(2\Omega t) + 2\cos^2(2\Omega t)] = \frac{32}{5}\frac{G}{c^5}\mu^2 L^4 \Omega^6 \,. \tag{14}$$

**B.2** 1.0pt

$$\xi = \frac{32}{5} \,.$$

**B.3** Now we assume a sequency of Keplerian orbits, with decreasing energy, which is being taken from the system by the GWs.

First, from (9), differentiating with respect to time,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{GM\mu}{2L^2}\frac{\mathrm{d}L}{\mathrm{d}t} \,, \tag{15}$$

Since this loss of energy is due to GWs, we equate it with (minus) the luminosity of GWs, given by (14)

$$\frac{GM\mu}{2L^2}\frac{\mathrm{d}L}{\mathrm{d}t} = -\frac{32}{5}\frac{G}{c^5}\mu^2 L^4 \Omega^6 \,. \tag{16}$$

We can eliminate the $L$ and $\mathrm{d}L/\mathrm{d}t$ dependence in this equation in terms of $\Omega$ and $\mathrm{d}\Omega/\mathrm{d}t$, by using Kepler's third law (8), which relates:

$$L^3 = G\frac{M}{\Omega^2} \,, \qquad \frac{\mathrm{d}L}{\mathrm{d}t} = -\frac{2}{3}\frac{L}{\Omega}\frac{\mathrm{d}\Omega}{\mathrm{d}t} \,. \tag{17}$$

Substituting in (16), we obtain:

$$\left(\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right)^3 = \left(\frac{96}{5}\right)^3 \frac{\Omega^{11}}{c^{15}} G^5 \mu^3 M^2 \equiv \left(\frac{96}{5}\right)^3 \frac{\Omega^{11}}{c^{15}} (GM_\mathrm{c})^5 \ . \tag{18}$$

| **B.3** | 1.0pt |
|---|---|
| $$M_\mathrm{c} = (\mu^3 M^2)^{1/5} \ .$$ | |

**B.4** Angular and cycle frequencies are related as $\Omega = 2\pi f$. From the information provided above: *GWs have a frequency which is twice as large as the orbital frequency*, we have

$$\frac{\Omega}{2\pi} = \frac{f_\mathrm{GW}}{2} \ . \tag{19}$$

Formula (10) of the question sheet has the form

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \chi \Omega^{11/3} \ , \qquad \chi \equiv \frac{96}{5} \frac{(GM_\mathrm{c})^{5/3}}{c^5} \ . \tag{20}$$

Thus, from (11) of the question sheet

$$\Omega(t)^{-8/3} = \frac{8}{3}\chi(t_0 - t) \ , \tag{21}$$

or, using (20) and the definition of $\chi$ gives

$$f_\mathrm{GW}^{-8/3}(t) = \frac{(8\pi)^{8/3}}{5} \left(\frac{GM_\mathrm{c}}{c^3}\right)^{5/3} (t_0 - t) \ . \tag{22}$$

| **B.4** | 2.0pt |
|---|---|
| $$p = 1 \ .$$ | |

**B.5** From the figure, we consider the two $\Delta t$'s as half periods. Thus, the (cycle) GW frequency is $f_\mathrm{GW} = 1/(2\Delta t)$. Then, the four given points allow us to compute the frequency at the mean time of the two intervals as

| | $t_\mathrm{AB}$ | $t_\mathrm{CD}$ |
|---|---|---|
| $t$ (s) | 0.0045 | 0.037 |
| $f_\mathrm{GW}$ (Hz) | $(2 \times 0.009)^{-1}$ | $(2 \times 0.006)^{-1}$ |

Now, using (22) we have two pairs of ($f_\mathrm{GW}$,$t$) values for two unknowns ($t_0$,$M_\mathrm{c}$). Expressing (22) for both $t_\mathrm{AB}$ and $t_\mathrm{CD}$ and dividing the two equations we obtain:

$$t_0 = \frac{At_\mathrm{CD} - t_\mathrm{AB}}{A - 1} \ , \qquad A \equiv \left(\frac{f_\mathrm{GW}(t_\mathrm{AB})}{f_\mathrm{GW}(t_\mathrm{CD})}\right)^{-8/3} \ . \tag{23}$$

Replacing by the numerical values, $A \simeq 2.95$ and $t_0 \simeq 0.054$ s. Now we can use (22) for either of the two values $t_\mathrm{AB}$ or $t_\mathrm{CD}$ and determine $M_\mathrm{c}$. One obtains for the chirp mass

$$M_\mathrm{c} \simeq 6 \times 10^{31} \ \mathrm{kg} \simeq 30 \times M_\odot \ . \tag{24}$$

Thus, the total mass $M$ is

$$M = 4^{3/5} M_\mathrm{c} \simeq 69 \times M_\odot \ . \tag{25}$$

This result is actually remarkably close to the best estimates using the full theory of General Relativity! [Even though the actual objects do not have precisely equal masses and the theory we have just used is not valid very close to the collision.]

| **B.5** | 1.0pt |
|---|---|
| $$M_{\text{c}} \simeq 30 \times M_{\odot} \,, \qquad M \simeq 69 \times M_{\odot} \,.$$ | |

**B.6** From (8), Kepler's law states that $L = (GM/\Omega^2)^{1/3}$. The second pair of points highlighted in the plot correspond to the cycle prior to merger. Thus, we use (19) to obtain the orbital angular velocity at $t_{\overline{\text{CD}}}$:

$$\Omega_{t_{\overline{\text{CD}}}} \sim 2.6 \times 10^2 \text{ rad/s} \,. \tag{26}$$

Then, using the total mass (25) we find

$$L \sim 5 \times 10^2 \text{ km} \,. \tag{27}$$

Thus, these objects have a maximum radius of $R_{\text{max}} \sim 250$ km. Hence they have over 30 times more mass and,

$$\frac{R_{\odot}}{R_{\text{max}}} \sim 3 \times 10^3 \tag{28}$$

they are 3000 times smaller than the Sun and!

Their linear velocity is

$$v_{\text{col}} = \frac{L}{2}\Omega \simeq 7 \times 10^4 \text{ km/s} \,. \tag{29}$$

They are moving at over 20% of the velocity of light!

| **B.6** | 1.0pt |
|---|---|
| $$L_{\text{collision}} \sim 5 \times 10^2 \text{ km} \,, \qquad \frac{R_{\odot}}{R_{\text{max}}} \sim 3 \times 10^3 \,, \qquad \frac{v_{\text{col}}}{c} \sim 0.2 \,.$$ | |

# Where is the neutrino? (10 points)

When two protons collide with a very high energy at the Large Hadron Collider (LHC), several particles may be produced as a result of that collision, such as electrons, muons, neutrinos, quarks, and their respective anti-particles. Most of these particles can be detected by the particle detector surrounding the collision point. For example, quarks undergo a process called *hadronisation* in which they become a shower of subatomic particles, called "jet". In addition, the high magnetic field present in the detectors allows even very energetic charged particles to curve enough for their momentum to be determined. The ATLAS detector uses a superconducting solenoid system that produces a constant and uniform 2.00 Tesla magnetic field in the inner part of the detector, surrounding the collision point. Charged particles with momenta below a certain value will be curved so strongly that they will loop repeatedly in the field and most likely not be measured. Due to its nature, the neutrino is not detected at all, as it escapes through the detector without interacting.

Data: Electron rest mass, $m = 9.11 \times 10^{-31}$ kg; Elementary charge, $e = 1.60 \times 10^{-19}$ C;

Speed of light, $c = 3.00 \times 10^{8}$ m s$^{-1}$; Vacuum permittivity, $\epsilon_0 = 8.85 \times 10^{-12}$ F m$^{-1}$

## Part A. ATLAS Detector physics (4.0 points)

**A.1** Derive an expression for the cyclotron radius, $r$, of the circular trajectory of an electron acted upon by a magnetic force perpendicular to its velocity, and express that radius as a function of its kinetic energy, $K$; charge modulus, $e$; mass, $m$; and magnetic field, $B$. Assume that the electron is a non-relativistic classical particle. 0.5pt

Electrons produced inside the ATLAS detector must be treated relativistically. However, the formula for the cyclotron radius also holds for relativistic motion when the relativistic momentum is considered.

**A.2** Calculate the minimum value of the momentum of an electron that allows it to escape the inner part of the detector in the radial direction. The inner part of the detector has a cylindrical shape with a radius of 1.1 meters, and the electron is produced in the collision point exactly in the center of the cylinder. Express your answer in MeV/$c$. 0.5pt

When accelerated perpendicularly to the velocity, relativistic particles of charge $e$ and rest mass $m$ emitt electromagnetic radiation, called synchrotron radiation. The emitted power is given by

$$P = \frac{e^2 a^2 \gamma^4}{6\pi\epsilon_0 c^3}$$

where $a$ is the acceleration and $\gamma = [1 - (v/c)^2]^{-1/2}$.

**A.3** A particle is called ultrarelativistic when its speed is very close to the speed of light. For an ultrarelativistic particle the emitted power can be expressed as: 1.0pt

$$P = \xi \frac{e^4}{\epsilon_0 m^k c^n} E^2 B^2 \,,$$

where $\xi$ is a real number, $n, k$ are integers, $E$ is the energy of the charged particle and $B$ is the magnetic field. Find $\xi$, $n$ and $k$.